Quasisymmetry Groups and Many-Body Scar Dynamics

Jie Ren,[1,2,*] Chenguang Liang,[1,2,*] and Chen Fang[1,3,4,†]

[1]*Beijing National Laboratory for Condensed Matter Physics and Institute of Physics, Chinese Academy of Sciences, Beijing 100190, China*

[2]*University of Chinese Academy of Sciences, Beijing 100049, China*

[3]*Songshan Lake Materials Laboratory, Dongguan, Guangdong 523808, China*

[4]*Kavli Institute for Theoretical Sciences, Chinese Academy of Sciences, Beijing 100190, China*

(Received 29 July 2020; revised 6 December 2020; accepted 24 February 2021; published 24 March 2021)

In quantum systems, a subspace spanned by degenerate eigenvectors of the Hamiltonian may have higher symmetries than those of the Hamiltonian itself. When this enhanced-symmetry group can be generated from local operators, we call it a quasisymmetry group. When the group is a Lie group, an external field coupled to certain generators of the quasisymmetry group lifts the degeneracy, and results in exactly periodic dynamics within the degenerate subspace, namely, the many-body-scar dynamics (given that Hamiltonian is nonintegrable). We provide two related schemes for constructing one-dimensional spin models having on-demand quasisymmetry groups, with exact periodic evolution of a prechosen product or matrix-product state under external fields.

DOI: 10.1103/PhysRevLett.126.120604

Introduction.—Symmetry plays a central role in physics. Given a quantum system described by Hamiltonian operator $\hat{H}$, a symmetry g, restricted to be unitary in this work, is represented by a unitary operator $\hat{D}(g)$, such that

$$[\hat{H}, \hat{D}(g)] = 0. \tag{1}$$

If multiple g's form a group G, Eq. (1) leads to the fundamental theorem that each eigensubspace $\Psi_E \equiv \{\psi | \hat{H}\psi = E\psi\}$ is invariant under $\hat{D}(g)$ for $g \in G$, or one can casually say that Ψ_E at least has symmetry group G. In other words, generally, Ψ_E has *higher* symmetry than G.

As an example, consider two $1/2$ spins coupled by a Heisenberg interaction, $\hat{H} = \hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2$. The full symmetry group of the triplet eigensubspace is U(3), of which the Hamiltonian symmetry group SO(3) is a subgroup. However, not all symmetries in U(3) are physically interesting, because many of them involve creating (annihilating) entanglement between the spins, and as such are difficult to realize in experiments. Therefore, hereafter we restrict to more physically relevant cases: an operator $\hat{D}(\tilde{g})$ that preserves an eigensubspace of $\hat{H}$ is considered as a "symmetry," if and only if $\hat{D}(\tilde{g})$ is a direct product of unitary operators on individual spins; that is,

$$\hat{D}(\tilde{g}) = \hat{d}_1(\tilde{g}) \otimes \hat{d}_2(\tilde{g}) \otimes \ldots \otimes \hat{d}_N(\tilde{g}), \tag{2}$$

known as the onsite-unitary condition. This requires the representation of G to be a tensor-product representation, that is, neither spatial nor time-reversal symmetry is considered, unless otherwise specified. In the above two-spin example, a unitary operation sending $|\uparrow\uparrow\rangle$ to $(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)/\sqrt{2}$ leaves the triplet eigensubspace invariant, but cannot decompose as in Eq. (2). In fact, it can be checked that all the symmetries of the triplet eigensubspace meeting the onsite-unitary condition Eq. (2) are just the overall rotations SO(3). The triplet eigensubspace has hence the same symmetry group as $\hat{H}$ itself.

The above discussion motivates us to define a new type of symmetry operation, which we tentatively term *quasisymmetry*, as a unitary operator $\hat{D}(\tilde{g})$ satisfying Eq. (2), so that a given eigensubspace of $\hat{H}$ having energy E is invariant under $\hat{D}(\tilde{g})$. It is obvious that $\tilde{g}$'s as such form a new group, denoted by $\tilde{G}_E$. We call $\tilde{G}_E$ the quasisymmetry group of $\hat{H}$ with respect to the eigensubspace Ψ_E. If $\hat{D}(\tilde{g})$ commutes with $\hat{H}$, then $\tilde{g}$ is a quasisymmetry for any eigensubspace of $\hat{H}$, so the symmetry group is always a subgroup of any quasisymmetry group for a given Hamiltonian: $G \subset \tilde{G}_E$.

Before showing an explicit example of quasisymmetry in quantum models, we point out that its classical counterpart, known as non-symmetry-caused degeneracy, is well known in models for frustrated magnetism. Consider a classical $J_1 - J_2$ model on a square lattice, where Heisenberg J_1 couplings connect nearest spins, and J_2 next-nearest-neighbor spins of length s. This Hamiltonian is invariant under any overall SO(3) rotation, but is *not* invariant under relative rotations between the two sublattices. Nevertheless, consider a state where all spins in each sublattice are antiferromagnetically aligned, then it is easy to check that the energy, being $-2J_2s^2$ per spin, is independent of the relative angle between the two sublattices. Therefore, a relative rotation between the sublattices, not being a symmetry of H, does lead to classical degeneracy. Can we obtain a quasisymmetry model by quantizing the above $J_1 - J_2$-model? The answer is negative: when quantum fluctuation is turned on, the above classical degeneracy is lifted due to the famous order-by-disorder mechanism [1].

We do not know a deterministic way for diagnosing all possible quasi-symmetries in a given Hamiltonian, quantum or classical. Yet fortunately, recent progress in the study on quantum-many-body scars [2–9] provides with many examples of quasisymmetry in quantum models [10]. In certain nonintegrable quantum many-body systems, there exist some close trajectories in the Hilbert space, along which a special short-range-entangled state evolves periodically or quasiperiodically, independent of the size of the system [12–18]. The evolution of certain many-body states along these closed trajectories, as opposed to the chaotic trajectories for generic states, is called the quantum-many-body scar dynamics, or simply scar dynamics. All the states along one such trajectory span a Hilbert subspace invariant under the Hamiltonian evolution, and the eigenstates of $\hat{H}$ within this subspace form a tower of states, namely, the scar tower [19–21]. The scar dynamics is related to the violation of the eigenstate-thermalization hypothesis [22–26] in certain eigenstates from the scar tower. In previously studied exact cases [27–33], a scar Hamiltonian consists of two parts,

$$\hat{H}_{\mathrm{scar}} = \hat{H} + \hat{H}_1, \tag{3}$$

where $\hat{H}$ has a degenerate eigensubspace Ψ_E and $\hat{H}_1$ (i) preserves the subspace Ψ_E but (ii) lifts the degeneracy by breaking energy spectrum into a "tower" with equal spacing δE. It then becomes obvious that a random initial state in Ψ_E oscillates with a period $2\pi\delta E^{-1}$. If a scar Hamiltonian in Eq. (3) satisfies (i) $\hat{H}_1$ is a sum of local operators and (ii) there is at least one product state $\psi_0 \in \Psi_E$, then the quantum Hamiltonian $\hat{H}$ has at least $\tilde{G} = \mathrm{U}(1)$ quasisymmetry $\hat{D}[\tilde{g}(\theta)] \equiv \exp(i\hat{H}_1\theta)$ with respect to Ψ_E. In other words, under the above conditions, quantum-many-body-scar dynamics is a sufficient condition for the existence of quasisymmetry.

Does quasisymmetry also imply scar dynamics? Suppose there is a quasisymmetry group $\tilde{G}_E \neq G$ for some $\hat{H}$ with respect to Ψ_E. If $\tilde{G}_E$ is a compact Lie group, then thanks to the onsite-unitary condition Eq. (2), we have that any generator

$$\hat{X} = \hat{x}_1 \oplus \hat{x}_2 \oplus \ldots \oplus \hat{x}_N \tag{4}$$

is a sum of local operators $\hat{x}_i$'s, each of which is a Hermitian operator acting on the ith spin. Choose $\hat{H}_1 = c\hat{X}$ for the scar Hamiltonian in Eq. (3), where c is a real constant. For any state $\psi(t=0) \in \Psi_E$ as initial state, we have

$$\hat{H}\psi(t) = \hat{H}\exp[-i(\hat{H}+\hat{H}_1)t]\psi(t=0) = E\psi(t), \tag{5}$$

meaning that Ψ_E is preserved by the scar Hamiltonian $\hat{H}_{\mathrm{scar}}$. Further, if X generates a U(1) subgroup of $\tilde{G}$, then the spectrum of $\hat{X}$ has equal spacing Δ, and the evolution of any $\psi \in \Psi_E$ has exact period $2\pi(c\Delta)^{-1}$. Therefore, quasisymmetry Lie group in $\hat{H}$ indeed implies scar dynamics, given that $\tilde{G}_E \neq G$. When $\tilde{G}_E$ is a discrete group, there is not an obvious choice for a scar Hamiltonian. In that case, there is a discrete version of scar dynamics, to be discussed in the Supplemental Material [34].

In this work, we focus on constructing spin Hamiltonians $\hat{H}$ that have a quasisymmetry group $\tilde{G}$ of choice. In the main text, we assume that the quasisymmetry group is a compact Lie group. Our construction scheme uses three elements as input: a spin-s spin chain defining the Hilbert space, $s = 1/2, 1, 3/2, \ldots$, a compact Lie group $\tilde{G}$ of choice, and an "anchor state," denoted by ψ_0, which is either a product or a matrix-product state [37]. For simplicity, we in this work only use two anchor states as examples: an all-up ferromagnetic state and an Affleck-Kennedy-Lieb-Tasaki-like [38] matrix-product state. The constructed Hamiltonian $\hat{H}$ is expressed in terms of projectors acting on small clusters, the same as in Ref. [39], but the method for defining the small-cluster projectors are based on two inputs: the anchor state and the quasisymmetry group [40].

Product states as anchor states.—We first describe the construction of spin-s Hamiltonians with a chosen $\tilde{G}$ using the all-up state $\psi_0 = |s\ldots s\rangle$ as the anchor state. To start with, we consider a cluster of m spins, or simply, an m cluster. The product state ψ_0 restricted to an m cluster is denoted by $\psi_0^{[m]}$. The unitary operators on a single spin form the unitary group $\mathrm{U}(2s+1)$, and we assume that $\tilde{G} \subset \mathrm{U}(2s+1)$. Define $\Psi_{\tilde{G}}^{[m]}$ as the following subspace in the m-cluster space

$$\Psi_{\tilde{G}}^{[m]} \equiv \mathrm{span}\{\hat{d}^{\otimes m}(\tilde{g})\psi_0^{[m]} | \tilde{g} \in \tilde{G}\}, \tag{6}$$

and define $\hat{P}$ as the projector onto $\Psi_{\tilde{G}}^{[m]}$. Then we consider the following m-cluster Hamiltonian

$$\hat{H}_{[m]} = (1-\hat{P})\hat{h}(1-\hat{P}), \tag{7}$$

where $\hat{h}$ is an arbitrary Hermitian matrix acting on the m cluster. It is easy to see that $\Psi_{\tilde{G}}^{[m]}$ is the zero energy subspace of $\hat{H}_{[m]}$ for a randomly chosen $\hat{h}$.

Now, consider an infinite chain. For each m cluster of consecutive spins we define a term as in Eq. (7), and obtain the full Hamiltonian

$$\hat{H} = \sum_{j=1,\ldots,N} (1-\hat{P}_{[j,j+m-1]})\hat{h}_{[j,j+m-1]}(1-\hat{P}_{[j,j+m-1]}), \tag{8}$$

where $\hat{P}_{[j,j+m-1]}$ is the m-cluster projector in Eq. (7) over the $j, j+1, \ldots, j+m-1$ spins, and $\hat{h}_{[j,j+m-1]}$ is a random Hermitian operator on the same cluster. The summation in Eq. (8) is from $j=1$ to $j=N-m+1$ if the chain is open, and to $j=N$ if closed. Periodic cycling is understood for a closed chain: when $j+l>N$, replace $j+l$ with $j+l-N$. Two observations can be made: (i) the all-up state ψ_0 is a zero-energy eigenstate of $\hat{H}$, because $(1-\hat{P}_{[j,j+m-1]})\psi_0 = 0$ for each j, and (ii) states of the following form

$$\hat{D}(\tilde{g})\psi_0 \equiv \hat{d}^{\otimes N}(\tilde{g})\psi_0 \tag{9}$$

are also zero-energy eigenstates of $\hat{H}$ for the same reason. All $\hat{D}(\tilde{g})\psi_0$'s in Eq. (9) and their linear combinations form a subspace $\Psi_{\tilde{G}} \equiv \mathrm{span}\{\hat{D}(\tilde{g})\psi_0 | \tilde{g} \in \tilde{G}\}$. It is clear that $\Psi_{\tilde{G}} \subset \Psi_0$, the zero energy subspace of $\hat{H}$. The Hamiltonian $\hat{H}$ hence has quasisymmetry group $\tilde{G}$ with respect to $\Psi_{\tilde{G}}$.

To better illustrate the scheme, we look at one example where $s=1$, $m=2$, and $\tilde{G} = \mathrm{SO}(3) \subset \mathrm{U}(3)$. For the 2-cluster, namely, the jth spin and the $(j+1)$th spin, the total spin $S = 0,\ 1,\ 2$, and the all-up state $\psi_0^{[2]} = |++\rangle$ belongs to $S=2$-subspace. Therefore acting $\hat{d}(\tilde{g}) \otimes \hat{d}(\tilde{g})$, where $\tilde{g} \in \mathrm{SO}(3)$ on $\psi_0^{[2]}$ yields the entire $S=2$-subspace, which is $\Psi_{\tilde{G}}^{[2]}$. The 2-cluster projector onto $\Psi_{\tilde{G}}^{[2]}$ is

$$\hat{P}_{[j,j+1]} = (\hat{\mathbf{S}}_j + \hat{\mathbf{S}}_{j+1})^2[(\hat{\mathbf{S}}_j + \hat{\mathbf{S}}_{j+1})^2 - 2]/24. \tag{10}$$

Substituting $\hat{P}_{[j,j+1]}$ and a random choice for $\hat{h}_{[j,j+1]}$ into Eq. (8), we have the full Hamiltonian. An exact diagonalization of this Hamiltonian (with periodic boundary) is carried out for $2 \le N \le 10$. We plot the level statistics in Ref. [34], which fits the Wigner-Dyson curve, indicating nonintegrability of the Hamiltonian [42]. The diagonalization also shows that there are exactly $2N+1$ independent states in Ψ_0, which are nothing but the states in the largest total spin sector (total spin being N), and that $\Psi_{\tilde{G}} = \Psi_0$.

We can also choose $\tilde{G} = \mathrm{SU}(2) \subset \mathrm{U}(3)$, and the same ψ_0 as the anchor state. In Ref. [34], we show that the resultant $\Psi_{\tilde{G}}$ (which again equals Ψ_0) is exactly spanned by, up to an onsite-unitary transform, the type-I scar tower of the spin-1-XY model in Ref. [27], although the Hamiltonian, due to the randomness in $\hat{h}_{[j,j+1]}$, can be drastically different from that of the XY model. (There are two scar towers discovered in Ref. [27], and we denote them, after their sequential appearances in the original paper, as type-I and type-II. Also see Ref. [29] for more on the type-II case.)

This simple example of the SO(3) quasisymmetry group illustrates some general features of quasisymmetry groups. First, $\tilde{G}$ is a subgroup of $\mathrm{U}(2s+1)$, so that by choosing a large s one can specify any compact Lie group, such as SO(n), U(n), Sp(n), and exceptional Lie groups, as the quasisymmetry group. We note here that the actual form of the "sandwiched" part of the Hamiltonian in Eq. (8), $\hat{h}_i$, is almost completely irrelevant, as long as it does not have so many symmetries that the Hamiltonian becomes integrable. Last, we want to emphasize that, despite the randomness in $\hat{h}_{[j,j+m-1]}$, it is *not* guaranteed that $\Psi_{\tilde{G}} = \Psi_0$. This indicates that the zero-energy subspace of $\hat{H}$, despite being designed to be so, is not generated by acting $\hat{D}(\tilde{G})$ on ψ_0. This equality between the two can only be established, or disproved, in numerics up to some N, as we do in Ref. [34].

Matrix-product states as anchor states.—A product state has zero entanglement, and if chosen as the anchor state, or, equivalently, the initial state, during the time evolution the state remains a product state, because quasisymmetry operations are strictly local. It is natural that we extend the discussion to the case where the anchor state has finite entanglement; i.e., is a matrix-product state. The corresponding construction of the scar Hamiltonian follows a slightly more complicated scheme, compared with the product-state case. Again considering a group $\tilde{G} \subset \mathrm{U}(2s+1)$, we first obtain two linear or projective representations of V of equal dimension χ, $d_L(\tilde{G})$, $d_R(\tilde{G})$, such that $d_L \otimes d_R$ contains a representation of dimension $2s+1$, denoted by $d(\tilde{G})$. In other words, there exists a trio of representations d_L, d_R, d of dimensions χ, χ, and $2s+1$, such that the Clebsch-Gordon coefficients $\langle d_L, \alpha; d_R, \beta | d, k\rangle \neq 0$, where $\alpha, \beta = 1, \ldots, \chi$ and $k = 1, \ldots, 2s+1$. When these conditions are met, define matrices (Fig. 1 shows how quasisymmetries act on these matrices)

$$A^k_{\alpha\beta} \equiv \langle d_L, \alpha; d_R, \beta | d, k\rangle. \tag{11}$$

These matrices define our anchor state(s), which is

$$\begin{aligned} \psi_0 &= \mathrm{Tr}(A^{s_1} \ldots A^{s_N}) | s_1, \ldots, s_N\rangle, \\ \psi_{\alpha\beta} &= (A^{s_1} \ldots A^{s_N})_{\alpha\beta} | s_1, \ldots, s_N\rangle, \end{aligned} \tag{12}$$

for a closed and an open chain, respectively.

Consider an m cluster, on which the matrices Eq. (11) define χ^2 open-matrix-product states

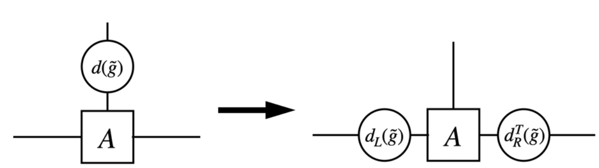

FIG. 1. Action of onsite operator $\hat{d}(\tilde{g})$ on the Clebsch-Gordon coefficients tensor A; the representation $d(\tilde{G})$ on the physical indices is transferred to two (projective) representations $d_{L,R}(\tilde{G})$ on the bond indices.

$$\psi_{\alpha\beta}^{[m]} = (A^{s_1} \ldots A^{s_m})_{\alpha\beta} |s_1, \ldots, s_m\rangle, \tag{13}$$

where $\alpha, \beta = 1, \ldots, \chi$. Acting $\hat{d}^{\otimes m}(\tilde{g})$ for any $\tilde{g} \in \tilde{G}$ on these χ^2 states yields another set of χ^2 open-matrix-product states:

$$\begin{aligned} \langle s_1 \ldots s_m | \hat{d}^{\otimes m}(\tilde{g}) \psi_{\alpha\beta}^{[m]} \rangle \\ &\equiv d_{s_1 s_1'}(\tilde{g}) \ldots d_{s_m s_m'}(\tilde{g}) [A^{s_1'} \ldots A^{s_m'}]_{\alpha\beta} \\ &= [d_L(\tilde{g}) A^{s_1} d_R^T(\tilde{g}) \ldots d_L(\tilde{g}) A^{s_m} d_R^T(\tilde{g})]_{\alpha\beta}. \end{aligned} \tag{14}$$

Find the subspace

$$\Psi_{\tilde{G}}^{[m]} \equiv \mathrm{span}\{\hat{d}^{\otimes m}(\tilde{g}) \psi_{\alpha\beta}^{[m]} | \tilde{g} \in \tilde{G}, \alpha, \beta = 1, \ldots, \chi\}, \tag{15}$$

and define $\hat{P}$ as the projector onto $\Psi_{\tilde{G}}^{[m]}$.

For a closed chain of $N \geq m$ sites, define the Hamiltonian as in Eq. (8). It is easy to verify that the anchor state ψ_0 is a zero eigenstate of $\hat{H}$ because it is a zero eigenstate of each term; and also the state $\hat{D}(\tilde{g})\psi_0 \equiv \hat{d}^{\otimes N}(\tilde{g})\psi_0$ is a zero eigenstate for the same reason for $\tilde{g} \in \tilde{G}$. The space $\Psi_{\tilde{G}}$ spanned by all these states is thus a zero-energy subspace of $\hat{H}$, i.e., $\Psi_{\tilde{G}} \subset \Psi_0$. Therefore, we have constructed $\hat{H}$ that has quasisymmetry group $\tilde{G}$ with respect to $\Psi_{\tilde{G}}$. The case of open chains can be similarly worked out (not shown here).

We again use an example to illustrate the above construction scheme. Choose $\tilde{G} = \mathrm{U}(1) \subset \mathrm{U}(3)$ as our quasigroup, and we choose $d_L(\tilde{G}) = d_R(\tilde{G}) = \frac{1}{2} \oplus -\frac{1}{2}$, which are the two-dimensional reducible projective representations of U(1). The specific realization of U(1) can be arbitrary, but in this example we choose it to be the overall spin rotation about the z axis. $d(\tilde{G})$ is chosen to be the three-dimensional reducible vector representation $d(\tilde{G}) = (x, y, z) = +1 \oplus 0 \oplus -1$. So the matrices are given by the Clebsch-Gordon coefficients

$$A^{\pm} = \sqrt{\frac{1}{6}}(\sigma_0 \pm \sigma_z), \qquad A^0 = \sqrt{\frac{1}{3}}\sigma_x, \tag{16}$$

satisfying

$$\exp(i\hat{S}_z\theta)_{ij} A^j = e^{i\sigma_z\theta/2} A^i (e^{i\sigma_z\theta/2})^T. \tag{17}$$

Now we consider an $m = 3$-cluster. The four open 3-cluster states, $\psi_{\alpha\beta}$, are none but the Affleck-Kennedy-Lieb-Tasaki open 3-chain ground states, up to a unitary transform $\exp(iS_y\pi)$ on all odd sites.

After acting all elements of the U(1) quasisymmetry group on the four open 3-cluster states, we have a subspace $\Psi_{\tilde{G}}^{[3]}$ spanned by 12 states, classified into groups labeled by two quantum numbers $n_{\pm} \equiv \hat{S}_{1z} \pm \hat{S}_{2z} + S_{3z}$:

$$\begin{aligned} (n_+, n_-) = (0,0)&: \frac{|+0-\rangle - |-0+\rangle}{\sqrt{2}}, \frac{|+0-\rangle + |-0+\rangle + |000\rangle}{\sqrt{3}}, \\ (\pm 1, \pm 1)&: \frac{|\pm 00\rangle + |00\pm\rangle}{\sqrt{2}}, \\ (\pm 1, \mp 1)&: |0 \pm 0\rangle, \\ (\pm 2, 0)&: |\pm\pm 0\rangle, |0 \pm\pm\rangle, \\ (\pm 3, \pm 1)&: |\pm\pm\pm\rangle. \end{aligned} \tag{18}$$

Define $\hat{P}$ as the 3-cluster projector onto $\Psi_{\tilde{G}}^{[3]}$. Replacing $\hat{P}_{[j,j+2]}$ with $\hat{P}$ in Eq. (8), we have the full Hamiltonian $\hat{H}$ with quasisymmetry U(1), with respect to the zero energy subspace $\Psi_{\tilde{G}}$. Using numerical calculation up to $N = 10$ sites [34], we find that level-spacing statistics of $\hat{H}$ shows Wigner Dyson behavior. We have also checked, up to $N = 14$, that the degeneracy of the zero subspace of $\hat{H}$ is $N + 1$ for periodic chains and $4N$ for open chains, and that $\Psi_0 = \Psi_{\tilde{G}}$. This means that the entire zero-energy subspace of $\hat{H}$ can be obtained from acting the quasisymmetry group elements on the anchor state(s). It is interesting to notice that, after an onsite-unitary transform, the resultant zero-energy subspace becomes the space spanned by the type-II-spin-1-XY scar [27,29]. We comment that since the quasisymmetry group is only U(1), instead of SU(2) or higher Lie groups, there is not an obvious choice for a local $\hat{Q}$ such that $[\hat{Q}, \hat{H}] = \mathrm{const} * \hat{Q}$ on the subspace. We also remark that the Hamiltonian following our construction is "unfrustrated," in the sense that Ψ_0 lies within the zero-energy subspace of each term in $\hat{H}$, in contrast to the original XY model. It is certainly possible to construct models having larger quasisymmetry groups, such as SO (3), using the same MPS as in Eq. (16), an explicit example of which is shown in Ref. [34].

We comment that using matrix-product states as anchor states is particularly useful when we relate this study to the study of symmetry-protected topological states [43–45] (SPT). In the Supplemental Material [34], we show how one can construct a scar tower and Hamiltonian such that all states of the form $\hat{D}(\tilde{g})\psi_0$ is an SPT protected by a unitary or antiunitary group. Here we simply point out that in the example above, both ψ_0 and $\hat{D}(\tilde{g})\psi_0$ are SPT protected by time-reversal symmetry, demonstrated in Ref. [34].

Discussion.—Aiming for a simple narrative, we have so far assumed that the anchor states have translation symmetry, and the quasi-group symmetry operator $\hat{D}(\tilde{g})$ acts uniformly on each spin, as in Eq. (9). Both conditions can be relaxed: (i) the anchor state may be rotated by onsite-unitary operators $\hat{d}_1(\tilde{g}_1) \otimes \hat{d}_2(\tilde{g}_2) \otimes \ldots \otimes \hat{d}_N(\tilde{g}_N)$ for $\tilde{g}_i \in \tilde{G}$; (ii) the action of $\hat{D}(\tilde{G})$ can be generalized to

$$\hat{D}(\tilde{g}) = \hat{d}_1(\tilde{g}) \otimes \hat{d}_2(\tilde{g}) \otimes \ldots \otimes \hat{d}_N(\tilde{g}), \qquad (19)$$

where $\hat{d}_{i=1,\ldots,N}$ are N different representations of $\tilde{G}$. With these generalizations, the method for defining the m-cluster projectors becomes slightly modified, shown in Ref. [34].

The anchor state, product, or matrix product, is a key input for our construction scheme. It ensures that within the zero-energy subspace of constructed Hamiltonian, there is at least one state that is a (matrix) product state. The anchor state can also be used as the initial state in the associated scar dynamics, and due to the onsite-unitary condition, all the states along the entire trajectory are (matrix) product states as the anchor state. In previous studies, the state used as the origin, from which the scar tower is obtained using ladder operators, is an exact eigenstate of the scar Hamiltonian, rather than an initial state for scar dynamics.

We impose the quasisymmetry group $\tilde{G}$ without requiring a ladder operator $\hat{Q}$. However, if $\tilde{G}$ is a non-Abelian Lie group, a ladder operator can always be found, because in that case $\mathrm{SO}(3) \subset \tilde{G}$, and SO(3) has ladder operator $\hat{Q} = \hat{L}_x - i\hat{L}_y$. For $\tilde{G} = \mathrm{U}(1)$, we have used one above example to show that even in the absence of Q, the zero-energy subspace of $\hat{H}$ forms a scar tower identical to the type-II-spin-1-XY scar tower. On the other hand, if $\tilde{G} \supset \mathrm{SO}(3)$, there are, in general, multiple ladder operators. For example, when $\tilde{G} = \mathrm{SU}(3) \supset \mathrm{SU}(2)$, there are three different ladder operators, corresponding to the three natural embeddings of SU(2) in SU(3). See Ref. [34] for an explicit model, and a general discussion on the relation between the non-Abelian quasisymmetry group and ladder operators.

To summarize, we show that many-body-scar towers have hidden group structures that we call quasisymmetry groups, and propose schemes for constructing local Hamiltonians that host any chosen Lie group as its quasisymmetry group. As application of the new concept, we show that (i) several known scar models can be unified, (ii) a scar model having three sets of ladder operators can be found (see Ref. [34]), and (iii) a discrete version of many-body scar is established by choosing a discrete quasi-symmetry group (see Ref. [34]).

C. F. thanks B. A. Bernevig for discussion and feedback, and thanks Huan He for the first introduction to the study of quantum many-body scars. C. L. and J. R. thank Shu Chen for his encouragement and support at the early stage of this project. C. F. and J. R. acknowledge support from Ministry of Science and Technology of China under Grant No. 2016YFA0302400, National Science Foundation of China under Grant No. 11674370, and Chinese Academy of Sciences under Grant No. XXH13506-202 and XDB33000000. C. L. acknowledges support from Ministry of Science and Technology of China under Grant No. 2016YFA0300600.

[*]These authors contributed equally to this work.
[†]cfang@iphy.ac.cn

[1] P. Chandra, P. Coleman, and A. I. Larkin, Phys. Rev. Lett. **64**, 88 (1990).
[2] E. J. Heller, Phys. Rev. Lett. **53**, 1515 (1984).
[3] H. Bernien, S. Schwartz, A. Keesling, H. Levine, A. Omran, H. Pichler, S. Choi, A. S. Zibrov, M. Endres, M. Greiner *et al.*, Nature (London) **551**, 579 (2017).
[4] C. J. Turner, A. A. Michailidis, D. A. Abanin, M. Serbyn, and Z. Papić, Nat. Phys. **14**, 745 (2018).
[5] C. J. Turner, A. A. Michailidis, D. A. Abanin, M. Serbyn, and Z. Papić, Phys. Rev. B **98**, 155134 (2018).
[6] S. Moudgalya, S. Rachel, B. A. Bernevig, and N. Regnault, Phys. Rev. B **98**, 235155 (2018).
[7] S. Moudgalya, N. Regnault, and B. A. Bernevig, Phys. Rev. B **98**, 235156 (2018).
[8] C.-J. Lin and O. I. Motrunich, Phys. Rev. Lett. **122**, 173401 (2019).
[9] V. Khemani, C. R. Laumann, and A. Chandran, Phys. Rev. B **99**, 161101(R) (2019).
[10] Quasisymmetry without extensive degeneracy in quantum models has appeared earlier in the literature. See Ref. [11], for example.
[11] B. Sriram Shastry and B. Sutherland, Physica (Amsterdam) **108B+C**, 1069 (1981).
[12] S. Choi, C. J. Turner, H. Pichler, W. W. Ho, A. A. Michailidis, Z. Papić, M. Serbyn, M. D. Lukin, and D. A. Abanin, Phys. Rev. Lett. **122**, 220603 (2019).
[13] W. W. Ho, S. Choi, H. Pichler, and M. D. Lukin, Phys. Rev. Lett. **122**, 040603 (2019).
[14] A. A. Michailidis, C. J. Turner, Z. Papić, D. A. Abanin, and M. Serbyn, Phys. Rev. X **10**, 011055 (2020).
[15] K. Bull, I. Martin, and Z. Papić, Phys. Rev. Lett. **123**, 030601 (2019).
[16] A. Hudomal, I. Vasić, N. Regnault, and Z. Papić, Commun. Phys. **3**, 99 (2020).
[17] A. M. Alhambra, A. Anshu, and H. Wilming, Phys. Rev. B **101**, 205107 (2020).
[18] C. J. Turner, J.-Y. Desaules, K. Bull, and Z. Papić, Correspondence Principle for Many-Body Scars in Ultracold Rydberg Atoms, arXiv:2006.13207 [Phys. Rev. X (to be published)].
[19] D. K. Mark, C.-J. Lin, and O. I. Motrunich, Phys. Rev. B **101**, 195131 (2020).
[20] S. Moudgalya, E. O'Brien, B. A. Bernevig, P. Fendley, and N. Regnault, Phys. Rev. B **102**, 085120 (2020).

[21] K. Bull, J.-Y. Desaules, and Z. Papić, Phys. Rev. B **101**, 165139 (2020).
[22] J. M. Deutsch, Phys. Rev. A **43**, 2046 (1991).
[23] M. Srednicki, Phys. Rev. E **50**, 888 (1994).
[24] M. Rigol, V. Dunjko, and M. Olshanii, Nature (London) **452**, 854 (2008).
[25] R. Nandkishore and D. A. Huse, Annu. Rev. Condens. Matter Phys. **6**, 15 (2015).
[26] L. D'Alessio, Y. Kafri, A. Polkovnikov, and M. Rigol, Adv. Phys. **65**, 239 (2016).
[27] M. Schecter and T. Iadecola, Phys. Rev. Lett. **123**, 147201 (2019).
[28] T. Iadecola and M. Schecter, Phys. Rev. B **101**, 024306 (2020).
[29] S. Chattopadhyay, H. Pichler, M. D. Lukin, and W. W. Ho, Phys. Rev. B **101**, 174308 (2020).
[30] N. Shibata, N. Yoshioka, and H. Katsura, Phys. Rev. Lett. **124**, 180604 (2020).
[31] D. K. Mark and O. I. Motrunich, Phys. Rev. B **102**, 075132 (2020).
[32] S. Moudgalya, N. Regnault, and B. A. Bernevig, Phys. Rev. B **102**, 085140 (2020).
[33] K. Pakrouski, P. N. Pallegar, F. K. Popov, and I. R. Klebanov, Phys. Rev. Lett. **125**, 230602 (2020).
[34] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.126.120604 for details regarding exact scar towers in quasi-symmetric models with (matrix-) product states as anchor states, discrete quasi-symmetries, inhomogeneous anchor states and group actions, non-Abelian quasi-symmetries, scar space of symmetry-protected topological states, and dynamics of scar states under generic perturbations, which includes Refs. [35,36].
[35] C.-J. Lin, A. Chandran, and O. I. Motrunich, Phys. Rev. Research **2**, 033044 (2020).
[36] E. H. Lieb and D. W. Robinson, Commun. Math. Phys. **28**, 251 (1972).
[37] F. Verstraete, V. Murg, and J. Cirac, Adv. Phys. **57**, 143 (2008).
[38] I. Affleck, T. Kennedy, E. H. Lieb, and H. Tasaki, Phys. Rev. Lett. **59**, 799 (1987).
[39] N. Shiraishi and T. Mori, Phys. Rev. Lett. **119**, 030601 (2017).
[40] There are other alternative ways to do the same job, for example, the inverse method in Ref. [41], but we choose the method that is least technical for better presentation.
[41] E. Chertkov and B. K. Clark, Phys. Rev. X **8**, 031029 (2018).
[42] A. Pal and D. A. Huse, Phys. Rev. B **82**, 174411 (2010).
[43] Z.-C. Gu and X.-G. Wen, Phys. Rev. B **80**, 155131 (2009).
[44] X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen, Science **338**, 1604 (2012).
[45] Y.-M. Lu and A. Vishwanath, Phys. Rev. B **86**, 125119 (2012).